Linearization of the Hamiltonian around the triangular equilibrium points in the generalized photogravitational Chermnykh's problem

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Abstract. Linearization of the Hamiltonian is being performed around the triangular equilibrium points in the generalized photogravitational Chermnykh's problem. The bigger primary is being considered as a source of radiation and small primary as an oblate spheroid. We have found the normal form of the second order part of the Hamiltonian. For this we have solved the aforesaid set of equations. . The effect of radiation pressure, gravitational potential from the belt on the linear stability have been examined analytically and numerically.

Keywords: Linearization, Chermnykh's problem, Radiation Pressure, Generalized Photogravitational, RTBP.

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1. Introduction

The Chermnykh's problem is new kind of restricted three body problem which was first time studied by (Chermnykh 1987). (Papadakis & Kanavos 2007) given numerical exploration of Chermnykh's problem, in which the equilibrium points and zero velocity curves studied numerically also the non-linear stability for the triangular Lagrangian points are computed numerically for the Earth-Moon and Sun-Jupiter mass distribution when the angular velocity varies. The mass reduction factor $q_1 = 1 - \frac{F_p}{F_g}$ is expressed in terms of the particle radius **a**, density ρ radiation pressure efficiency factor χ (in C.G.S. system):$q_1 = 1 - \frac{5.6\times10^{-5}}{\mathbf{a}\rho}\chi$. Where F_p is solar radiation pressure force which is exactly apposite to the gravitational attraction force F_g and change with the distance by the same law it is possible to consider that the result of action of this force will lead to reducing the effective mass of the Sun or particle.

(Ishwar & Kushvah 2006) examined the linear stability of triangular equilibrium points in the generalized photogravitational restricted three body problem with Poynting-Robertson drag, L_4 and L_5 points became unstable due to P-R drag which is very remarkable and important, where as they are linearly stable in classical problem when $0 < \mu < \mu_{Routh} = 0.0385201$. Further the normalizations of Hamiltonian and nonlinear stability of $L_{4(5)}$ in the present of P-R drag has been studied by (Kushvah et al. 2007a, Kushvah et al. 2007b, Kushvah et al. 2007c)

Normal forms are a standard tool in Hamiltonian mechanics to study the dynamics in a neighbourhood of invariant objects. Usually, these normal forms are obtained as divergent series, but their asymptotic character is what makes them useful. From theoretical point of view, they provide nonlinear approximations to the dynamics in a neighbourhood of the invariant object, that allows to obtain information about the real solutions of the system by taking the normal form upto a suitable finite order. In this case, it is well known that under certain(generic) non-resonance conditions, the remainder of this finite normal form turns out to be exponentially small with respect to some parameters. Those series are usually divergent on open sets, it is still possible in some cases to prove convergence on certain sets with empty interior(Cantor-like sets) by replacing the standard linear normal form scheme by a quadratic one.

From a more practical point of view, normal forms can be used as a computational method to obtain very accurate approximations to the dynamics in a neighbourhood of the selected invariant object, by neglecting the remainder. They have been applied, for example, to compute invariant manifolds or invariant tori. To do that, it is necessary to compute the explicit expression of the normal form and of the (canonical) transformation that put the Hamiltonian into this reduced form. A context where this computational formulation has special interest is in some celestial mechanics models, that can be used to approximate the dynamics of some real world problems.

The linearization of Hamiltonian, have been studied numerically and analytically. We

have examined the effect of gravitational potential from the belt, oblateness effect and radiation effect.

2. Equations of Motion and Position of Equilibrium Points

Let us consider the model proposed by (Miyamoto & Nagai 1975), according to this model the potential of belt is given by:

$$V(r,z) = \frac{\mathbf{b}^2 M_b \left[\mathbf{a}r^2 + (\mathbf{a}+3N)\right](\mathbf{a}+N)^2}{N^3\left[r^2+(\mathbf{a}+N)^2\right]^{5/2}} \tag{1}$$

where M_b is the total mass of the belt and $r^2 = x^2+y^2$, $\mathbf{a}, \mathbf{b}$ are parameters which determine the density profile of the belt, if $\mathbf{a}=\mathbf{b}=0$ then the potential equals to the one by a point mass. The parameter $\mathbf{a}$ "flatness parameter"and $\mathbf{b}$ "core parameter", where $N=\sqrt{z^2+\mathbf{b}^2}$, $T=\mathbf{a}+\mathbf{b}$, $z=0$. Then we obtained

$$V(r,0) = -\frac{M_b}{\sqrt{r^2+T^2}} \tag{2}$$

and $V_x = \frac{M_b x}{(r^2+T^2)^{3/2}}$, $V_y = \frac{M_b y}{(r^2+T^2)^{3/2}}$. As in (Kushvah 2008$a$, Kushvah 2008$b$), we consider the barycentric rotating co-ordinate system $Oxyz$ relative to inertial system with angular velocity ω and common z–axis. We have taken line joining the primaries as x–axis. Let m_1, m_2 be the masses of bigger primary(Sun) and smaller primary(Earth) respectively. Let Ox, Oy in the equatorial plane of smaller primary and Oz coinciding with the polar axis of m_2. Let r_e, r_p be the equatorial and polar radii of m_2 respectively, r be the distance between primaries. Let infinitesimal mass m be placed at the point $P(x,y,0)$. We take units such that sum of the masses and distance between primaries as unity, the unit of time taken such that the Gaussian constant of gravitational $\Bbbk^2=1$. Then perturbed mean motion n of the primaries is given by $n^2 = 1+\frac{3A_2}{2}+\frac{2M_b r_c}{(r_c^2+T^2)^{3/2}}$, where $r_c^2=(1-\mu)q_1^{2/3}+\mu^2$, $A_2=\frac{r_e^2-r_p^2}{5r^2}$ is oblateness coefficient of m_2. where $\mu=\frac{m_2}{m_1+m_2}$ is mass parameter, $1-\mu=\frac{m_1}{m_1+m_2}$ with $m_1>m_2$. Then coordinates of m_1 and m_2 are $(x_1,0)=(-\mu,0)$ and $(x_2,0)=(1-\mu,0)$ respectively. The mass of Sun $m_1 \approx 1.989\times10^{30}kg \approx 332,946m_2$(The mass of Earth), hence mass parameter for this system is $\mu=3.00348\times10^{-6}$. In the above mentioned reference system, we determine the equations of motion of the infinitesimal mass particle in xy-plane. Now using (Miyamoto & Nagai 1975) profile and (Kushvah 2008a, Kushvah 2008b), the equations of motion are given by:

$$\ddot{x}-2n\dot{y} = U_x - V_x = \Omega_x, \tag{3}$$

$$\ddot{y}+2n\dot{x} = U_y - V_y = \Omega_y \tag{4}$$

where

$$\Omega_x = n^2x - \frac{(1-\mu)q_1(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3} - \frac{3}{2}\frac{\mu A_2(x+\mu-1)}{r_2^5} - \frac{M_b x}{(r^2+T^2)^{3/2}},$$

$$\Omega_y = n^2 y - \frac{(1-\mu)q_1 y}{r_1^3} - \frac{\mu y}{r_2^3} - \frac{3}{2}\frac{\mu A_2 y}{r_2^5} - \frac{M_b y}{(r^2+T^2)^{3/2}},$$
$$\Omega = \frac{n^2(x^2+y^2)}{2} + \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} + \frac{M_b}{(r^2+T^2)^{1/2}} \quad (5)$$

The energy integral of the problem is given by $C = 2\Omega - \dot{x}^2 - \dot{y}^2$, where the quantity C is the Jacobi's constant. The zero velocity curves $C = 2\Omega(x, y)$ are presented in figure (1) for the entire range of parametersA_2, M_b and $q_1 = 1, 0$. We have seen that there are closed curves around the $L_{4(5)}$ so they are stable but the stability range reduced(2)] due radiation effect. The closed curves around $L_{4(5)}$ disappeared when $q_1 = 0$.

The position of equilibrium points are given by putting $\Omega_x = \Omega_y = 0$ i.e.,

$$n^2 x - \frac{(1-\mu)q_1(x+\mu)}{r_1^3} - \frac{\mu(x+\mu-1)}{r_2^3}$$
$$-\frac{3}{2}\frac{\mu A_2(x+\mu-1)}{r_2^5} - \frac{M_b x}{(r^2+T^2)^{3/2}} = 0, \quad (6)$$
$$n^2 y - \frac{(1-\mu)q_1 y}{r_1^3} - \frac{\mu y}{r_2^3} - \frac{3}{2}\frac{\mu A_2 y}{r_2^5}$$
$$-\frac{M_b y}{(r^2+T^2)^{3/2}} = 0 \quad (7)$$

from equations (6, 7) we obtained:

$$r_1 = q_1^{1/3}\left[1 - \frac{A_2}{2} + \frac{(1-2r_c)M_b\left(1 - \frac{3\mu A_2}{2(1-\mu)}\right)}{3\left(r_c^2+T^2\right)^{3/2}}\right], \quad (8)$$
$$r_2 = 1 + \frac{\mu(1-2r_c)M_b}{3\left(r_c^2+T^2\right)^{3/2}} \quad (9)$$

From above, we obtained:

$$x = -\mu \pm \left[\left(\frac{q_1}{n^2}\right)^{2/3}\left[1 + \frac{3A_2}{2}\right.\right.$$
$$\left.\left.-\frac{(1-2r_c)M_b\left(1-\frac{3\mu A_2}{2(1-\mu)}\right)}{\left(r_c^2+T^2\right)^{3/2}}\right]^{-2/3} - y^2\right]^{1/2} \quad (10)$$
$$x = 1 - \mu$$
$$\pm\left[\left[1 - \frac{\mu(1-2r_0)M_b}{\left(r_c^2+T^2\right)^{3/2}}\right]^{-2/3} - y^2\right]^{1/2} \quad (11)$$

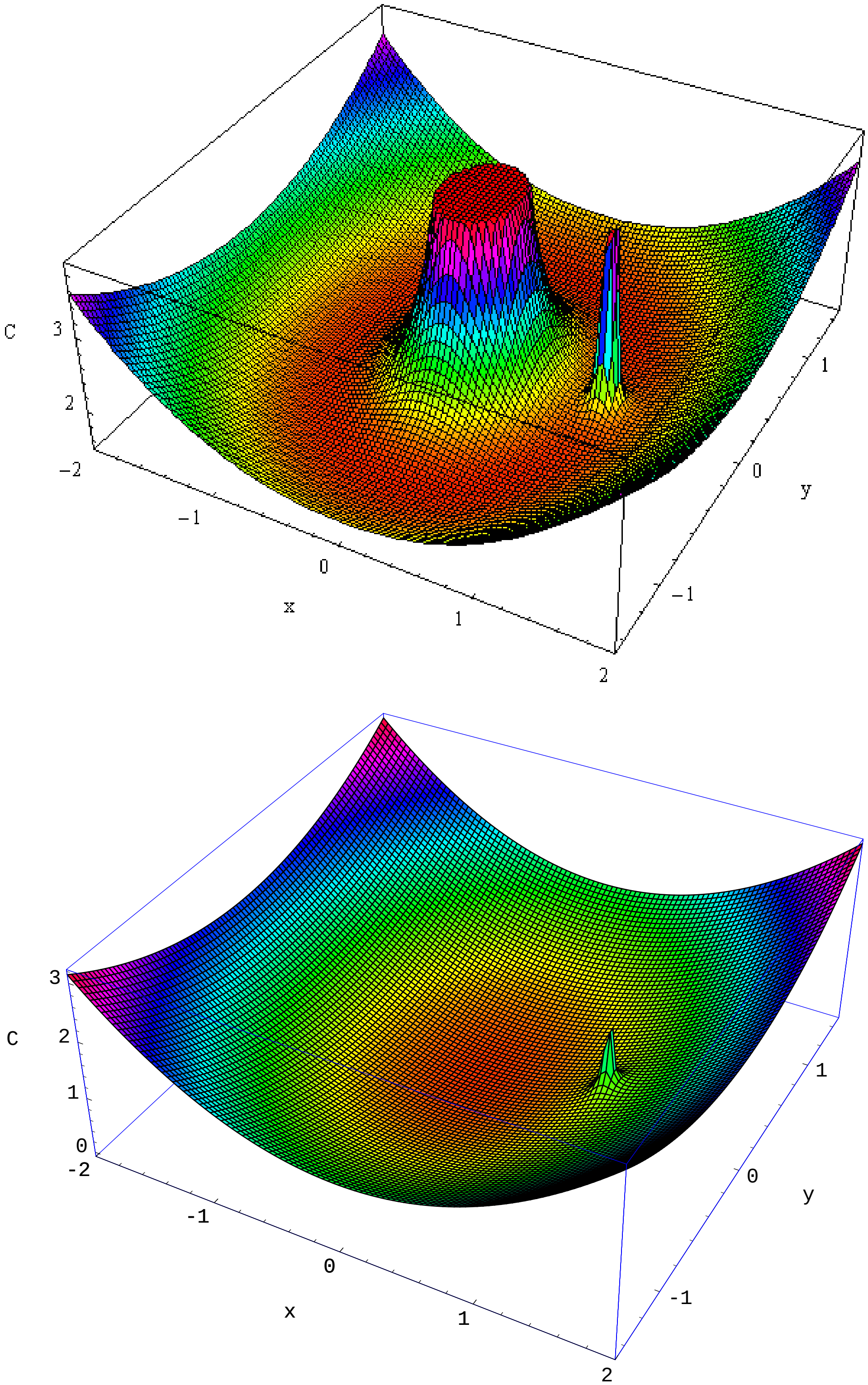

Figure 1. The zero velocity surfaces for $\mu = 0.025, r_c = 0.9999, T = 0.01$, for all values of A_2, M_b

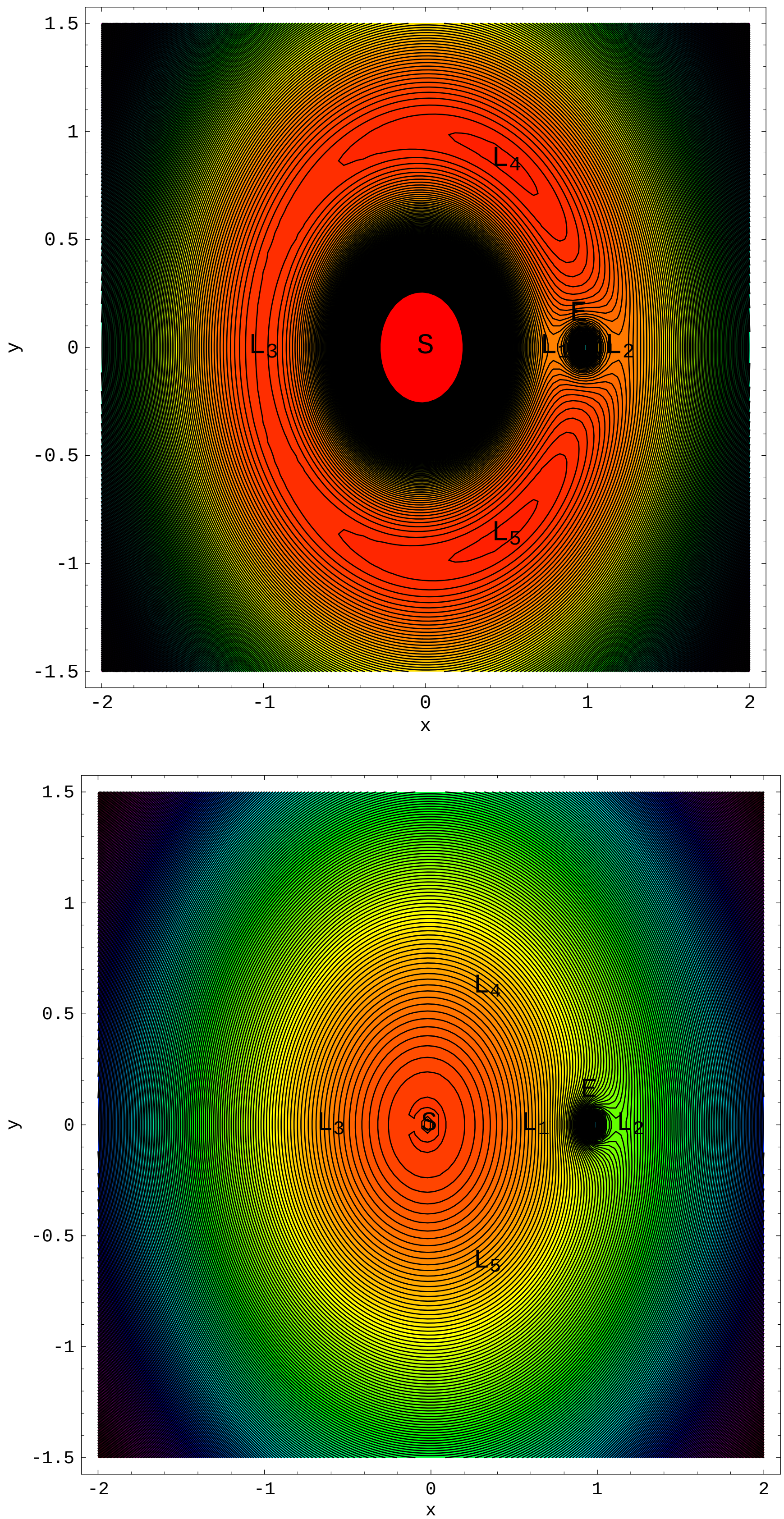

Figure 2. The figure show the zero velocity curves for $\mu = 0.025, T = 0.01$, in frame one $q_1 = 0.00$, in second frame show the curves in classical case.

The triangular equilibrium points are given by putting $\Omega_x = \Omega_y = 0$, $y \neq 0$, then from equations (3) and (4) we obtained the triangular equilibrium points as:

$$x = -\mu + \frac{q_1^{2/3}}{2}(1 - A_2) + \frac{(1-2r_c)M_b\left[\left\{1 - \frac{3\mu A_2}{(1-\mu)}\right\}q_1^{2/3} - 1\right]}{3\left(r_c^2 + T^2\right)^{3/2}} \tag{12}$$

$$y = \pm\frac{q_1^{2/3}}{2}\left[4 - q_1^{2/3} + 2\left(q_1^{2/3} - 2\right)A_2 \right.$$
$$\left. - \frac{4(2r_c - 1)M_b\left[\left\{\left(q_1^{2/3} - 3\right) - \frac{3\mu A_2\left(q_1^{2/3}-3\right)}{2(1-\mu)}\right\}\right]}{3\left(r_c^2 + T^2\right)^{3/2}}\right]^{1/2} \tag{13}$$

All these results are similar with (Szebehely 1967), (Ragos & Zafiropoulos 1995), (Kushvah 2008*a*, Kushvah 2008*b*) and others.

3. Linearization of the Hamiltonian

We have to expand the Lagrangian function in power series of x and y, where (x, y) are the coordinates of the triangular equilibrium points. We will examine the stability of the triangular equilibrium points. For this we will utilize the method of (Whittaker 1965). By taking H_2, we will consider linear equations this we have established the relations between perturbed basic frequencies. The Lagrangian function of the problem can be written as

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2 + y^2)$$
$$+ \frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} + \frac{M_b}{\left(r^2 + T^2\right)^{1/2}} \tag{14}$$

and the Hamiltonian $H = -L + p_x\dot{x} + p_y\dot{y}$, where p_x, p_y are the momenta coordinates given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - ny, \quad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + nx$$

Let us suppose $q_1 = 1 - \epsilon$, with $|\epsilon| << 1$ and rewriting the coordinates of triangular equilibrium points as:

$$x = \frac{\gamma}{2} - \frac{\epsilon}{3} - \frac{A_2}{2} + \frac{A_2\epsilon}{3} + \frac{2M_b\epsilon}{9} - \frac{\mu A_2 M_b}{2(1-\mu)}\left(1 - \frac{2}{3}\epsilon\right) \tag{15}$$

$$y = \pm\frac{\sqrt{3}}{2}\left\{1 - \frac{5\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} - \frac{4M_b}{9} - \frac{8M_b\epsilon}{27} + \frac{\mu A_2 M_b\epsilon}{9(1-\mu)}\right\} \tag{16}$$

where $\gamma = 1 - 2\mu$. We shift the origin to L_4, change $x \to x_* + x$, $y \to y_* + y$. and $a = x_* + \mu, b = y_*$,then

$$a = \frac{1}{2}\left\{1 - \frac{2\epsilon}{3} - A_2 + \frac{2A_2\epsilon}{3} + \frac{4M_b\epsilon}{9} - \frac{\mu A_2 M_b}{(1-\mu)}\left(1 - \frac{2}{3}\epsilon\right)\right\} \tag{17}$$

$$b = \frac{\sqrt{3}}{2}\left\{1 - \frac{2\epsilon}{9} - \frac{A_2}{3} - \frac{2A_2\epsilon}{9} - \frac{4M_b}{9} - \frac{8M_b\epsilon}{27} + \frac{\mu A_2 M_b \epsilon}{9(1-\mu)}\right\} \tag{18}$$

We have to expand L in power series of x and y, for this we use Taylor's expansion i.e.

$$\begin{aligned} f(x,y) &= f(0,0,) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2}[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)] \\ &+ \frac{1}{3!}[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \ldots \end{aligned} \tag{19}$$

Using these values in (19) and with the help of (17) and (18) we get

$$L = L_0 + L_1 + L_2 + L_3 + \cdots \tag{20}$$

$$H = H_0 + H_1 + H_2 + H_3 + \cdots = -L + p_x\dot{x} + p_y\dot{y} \tag{21}$$

where $L_0, L_1, L_2, \ldots$ are

$$\begin{aligned} L_0 = &\left[(a-\mu)^2 + b^2\right] n^2 + \frac{(1-\mu)q_1}{\sqrt{a^2+b^2}} \\ &+ \frac{\mu}{\sqrt{(a-1)^2+b^2}}\left\{1 + \frac{A_2}{2(a-1)^2+b^2}\right\} + \frac{M_b}{((a-\mu)^2+b^2+T^2)^{3/2}} \end{aligned} \tag{22}$$

$$\begin{aligned} L_1 = x &\left[2n^2(a-\mu) - \frac{a(1-\mu)q_1}{(a^2+b^2)^{3/2}} - \frac{3(a-\mu)M_b}{((a-\mu)^2+b^2+T^2)^{5/2}}\right. \\ &\left. - \frac{(a-1)\mu}{(a-1)^2+b^2)^{3/2}}\left\{1 + \frac{3A_2}{2(a-1)^2+b^2}\right\}\right] \\ y &\left[2bn^2 - \frac{b(1-\mu)q_1}{(a^2+b^2)^{3/2}} - \frac{3bM_b}{((a-\mu)^2+b^2+T^2)^{5/2}}\right. \\ &\left. - \frac{b\mu}{(a-1)^2+b^2)^{3/2}}\left\{1 + \frac{3A_2}{2(a-1)^2+b^2}\right\}\right] \end{aligned} \tag{23}$$

$$L_2 = \frac{(\dot{x}^2+\dot{y}^2)}{2} + n(x\dot{y} - \dot{x}y) + \frac{n^2}{2}(x^2+y^2) - Ex^2 - Fy^2 - Gxy \tag{24}$$

$$\begin{aligned} E = \frac{1}{1728} &\left[2 - 324(2-25\mu)M_b - \frac{(2r_c-1)\{2+15(2-7\mu)M_b\}M_b}{(r_c^2+T^2)^{3/2}}\right. \\ &+ 2\epsilon\left\{-54 - 270\mu + 135(10-81\mu)M_b \right. \\ &\left. - \frac{(2r_c-1)\{146-33\mu+15(118-205\mu)M_b\}M_b}{(r_c^2+T^2)^{3/2}}\right\} \\ &+ 6A_2\left\{162 - 432\mu + 135(2+39\mu)M_b \right. \\ &\left. + \frac{(2r_c-1)\{146-240\mu-15(10-259\mu)M_b\}M_b}{(r_c^2+T^2)^{3/2}}\right\} \end{aligned}$$

$$+ \epsilon A_2 \{144 - 5022\mu - 270(4 - 395\mu)M_b$$

$$\left. \left. - \frac{(2r_c - 1)\{458 + 540\mu + 15(1926 - 7399\mu)M_b\} M_b}{(r_c^2 + T^2)^{3/2}} \right\} \right] \tag{25}$$

$$F = \frac{-1}{576} \left[360 + 108(22 + 85\mu)M_b + \frac{(2r_c - 1)\{2 + 5(6 + 35\mu)M_b\} M_b}{(r_c^2 + T^2)^{3/2}} \right.$$

$$+ 2\epsilon \{54 - 18\mu + 45(62 + 309\mu)M_b$$

$$\left. - \frac{(2r_c - 1)\{10 + 39\mu + 5(202 + 1565\mu)M_b\} M_b}{(r_c^2 + T^2)^{3/2}} \right\}$$

$$+ 6A_2 \{126 + 45(26 + 99\mu)M_b$$

$$\left. + \frac{(2r_c - 1)\{46 + 5(142 + 791\mu)M_b\} M_b}{(r_c^2 + T^2)^{3/2}} \right\}$$

$$+ \epsilon A_2 \{576 - 18\mu + 90(228 + 1147\mu)M_b$$

$$\left. \left. + \frac{(2r_c - 1)\{522 + 526\mu + 5(3834 + 27923\mu)M_b\} M_b}{(r_c^2 + T^2)^{3/2}} \right\} \right] \tag{26}$$

$$G = \frac{-1}{288\sqrt{3}} [648 - 1296\mu + 1620(2 + 3\mu)M_b$$

$$+ \frac{12(2r_c - 1)\{22 - 44\mu + 5(34 + 39\mu)M_b\} M_b}{(r_c^2 + T^2)^{3/2}}$$

$$+ 2\epsilon \{198 - 666\mu + 45(94 + 201\mu)M_b$$

$$\left. + \frac{2(2r_c - 1)\{190 - 423\mu + 135(18 + 65\mu)M_b\} M_b}{(r_c^2 + T^2)^{3/2}} \right\}$$

$$+ 6A_2 \{126 - 468\mu + 45(26 + 15\mu)M_b$$

$$\left. + \frac{(2r_c - 1)\{96 - 260\mu + 15(66 + 133\mu)M_b\} M_b}{(r_c^2 + T^2)^{3/2}} \right\}$$

$$+ \epsilon A_2 \{1368 - 4806\mu + 90(280 + 429\mu)M_b$$

$$\left. \left. + \frac{(2r_c - 1)\{2106 - 655\mu + 35(982 + 3025\mu)M_b\} M_b}{(r_c^2 + T^2)^{3/2}} \right\} \right] \tag{27}$$

3.1. Perturbed Basic Frequencies

Using the method in (Whittaker 1965), to find the canonical transformation from the phase space (x, y, p_x, p_y) into the phase space product of the angle co-ordinates (ϕ_1, ϕ_2) and the action momenta co-ordinates I_1, I_2 and of the first order in $I_1^{1/2}, I_2^{1/2}$. We consider the

following linear equations in the variables x, y:

$$\begin{array}{rclrcl} -\lambda p_x & = & \frac{\partial H_2}{\partial x}, & \lambda x & = & \frac{\partial H_2}{\partial p_x}, \\ -\lambda p_y & = & \frac{\partial H_2}{\partial y}, & \lambda y & = & \frac{\partial H_2}{\partial p_y}, \\ AX & = & 0 & & & \end{array} \tag{28}$$

$$X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix} \quad and \quad A = \begin{bmatrix} 2E & G & \lambda & -n \\ G & 2F & n & \lambda \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{bmatrix} \tag{29}$$

The characteristic equation of the Hamiltonian

$$H_2 = \frac{p_x^2 + p_y^2}{2} + n(yp_x - xp_y) + Ex^2 + Fy^2 + Gxy$$

is given by $|A| = 0$ i.e.

$$\lambda^4 + 2(E + F + n^2)\lambda^2 + 4EF - G^2 + n^4 - 2n^2(E + F) = 0 \tag{30}$$

Stability is assured only when the discriminant $D > 0$, where

$$D = 4(E + F + n^2)^2 - 4\{4EF - G^2 + n^4 - 2n^2(E + F)\} \tag{31}$$

from stability condition we obtained:

$$\begin{aligned} \mu < \mu_{c_0} &+ 0.125885\epsilon + M_b \{0.571136 + 1.73097\epsilon \\ &+ \frac{0.219964 - 0.398363\epsilon + r_c(0.202129 + 0.114305\epsilon)}{(r_c^2 + T^2)^{3/2}} \} \\ &- [0.0627796 - 0.112691\epsilon + M_b A_2 \{0.281354 - 1.53665\epsilon \\ &+ \frac{0.654936 - 0.669428\epsilon + r_c(0.195486 + 0.350878\epsilon)}{(r_c^2 + T^2)^{3/2}} \Big\} \Big] \end{aligned} \tag{32}$$

where $\mu_{c_0} = 0.0385209$. When $D > 0$ the roots $\pm i\omega_1$ and $\pm i\omega_2$ (ω_1, ω_2 being the long/short -periodic frequencies) are related to each other as

$$\begin{aligned} \omega_1^2 + \omega_2^2 = &\frac{1}{54} [27((1 + \mu)\epsilon - 2) + 9[-18 + 36\mu + (22 + 69\mu)] \\ &+ \frac{81M_b}{2}(12 + 30\epsilon + 30\mu + 95\mu\epsilon) + 135M_bA_2(18 + 58\epsilon + 45\mu + 188\mu\epsilon) \\ &+ \frac{1}{2\,(r_c^2 + T^2)^{3/2}} \Big\{ M_b\,(180 + 2(2r_c - 1)(44 + 21)\mu\epsilon \\ &+ 72r_c + 4r_cA_2[78 + 180\mu + (253 + 873\mu)\epsilon]) \Big\} \Big] \end{aligned} \tag{33}$$

$$
\begin{aligned}
\omega_1^2\omega_2^2 = &-\frac{1}{72}\left[108[\epsilon+3M_b(2+7\epsilon)]+18[31\epsilon+M_b(144+647\epsilon)]A_2\right.\\
&+\left.\frac{M_b\left\{4[36-47\epsilon+4r_c(9+28\epsilon)]+3[74-297\epsilon+4r_c(44+175\epsilon)]A_2\right\}}{(r_c^2+T^2)^{3/2}}\right]\\
&+\mu\left[\frac{27}{4}+\frac{99\epsilon}{8}+\frac{117}{4}+73A_2\epsilon+M_b\left\{\frac{45}{4}+81A_2+\frac{273\epsilon}{8}+\frac{2357A_2\epsilon}{8}\right.\right.\\
&+\left.\left.\frac{396(2r_c-1)+4(772r_c-395)\epsilon+12(326r_c-181)A_2+(18452r_c-9667)A_2\epsilon}{72\,(r_c^2+T^2)^{3/2}}\right\}\right]\\
&+\mu^2\left[\frac{-27}{4}+\frac{111\epsilon}{8}+\frac{117}{4}+\frac{161A_2\epsilon}{2}+M_b\left\{\frac{405}{4}+\frac{495A_2}{2}+\frac{4185\epsilon}{16}+\frac{25275A_2\epsilon}{16}\right.\right.\\
&+\left.\left.\frac{(2r_c-1)[198-838\epsilon-1014A_2-5117A_2\epsilon]}{36\,(r_c^2+T^2)^{3/2}}\right\}\right] \qquad (34)
\end{aligned}
$$

where $\omega_j(j=1,2)$ satisfy the property $(0<\omega_2<\frac{1}{\sqrt{2}}<\omega_1<1)$. The mass parameter is shown in figure $0\leq M_b\leq 2$, $0\leq A_2\leq 1$, surface $(1)q_1=1$ (2) $q_1=0.75$, $(3)q_1=0.5$,$(4)q_1=0.25$.

3.2. The Normal Coordinates

For expressing H_2 in a simpler form, we consider the set of linear equations(28) the solution of which can be obtained as

$$
\begin{aligned}
\frac{x}{(2n\lambda-G)} = \frac{y}{(\lambda^2-n^2+2E)} &= \frac{p_x}{(n\lambda^2-G\lambda-2nE+n^3)}\\
&= \frac{p_y}{(\lambda^3+n^2\lambda+2E\lambda-nG)} \qquad (35)
\end{aligned}
$$

Substituting $\lambda=\pm i\omega_1$ and $\pm i\omega_2$, we obtain the solution sets as

$$
\begin{aligned}
x_j &= K_j(2ni\omega_j-G), \quad p_{x,j}=K_j(-n\omega_j^2-iG\omega_j-2En+n^3)\\
y_j &= K_j(-\omega_j-n^2+2E), \quad p_{y,j}=K_j\left\{-i\omega_j^3+i\omega_j(n^2+2E)-Gn\right\}\\
x_{j+2} &= K_{j+2}(2ni\omega_j-G), \quad p_{x,j+2}=K_{j+2}(-n\omega_j^2+iG\omega_j-2En+n^3)\\
y_{j+2} &= K_{j+2}(-\omega_j^2-n^2+2E), \quad p_{y,j+2}=K_{j+2}\left\{\omega_j^3-i\omega_j(n^2+2E)-Gn\right\}
\end{aligned}
$$

where $j=1,2$ and K_j,K_{j+2} are constants of proportionality. Following the method for reducing H_2 to the normal form, as in (Whittaker 1965), use the transformation

$$
X=JT \quad where \quad X=\begin{bmatrix} x\\ y\\ p_x\\ p_y\end{bmatrix}, \quad T=\begin{bmatrix} Q_1\\ Q_2\\ P_1\\ P_2\end{bmatrix} \qquad (36)
$$

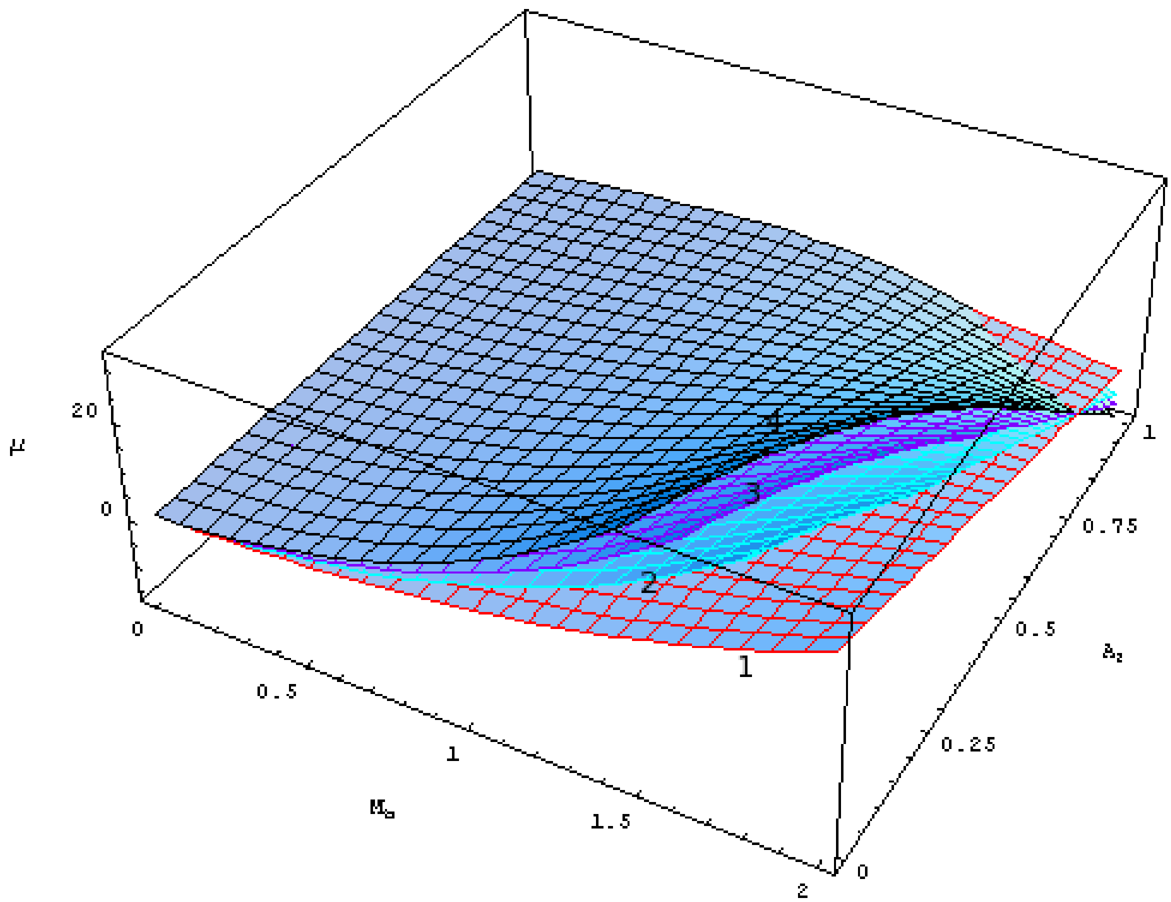

Figure 3. The critical mass ratios in the 3D plot for $M_b - A_2 - \mu$ and $q_1 = 1, 0.75, 0.5, 0.25$, $r_c = 0.9999, T = 0.01$

and $J = [J_{ij}]_{1 \leq i,j \leq 4}$ i.e.

$$J = \begin{bmatrix} x_1 - i\frac{\omega_1 x_3}{2} & -x_2 - i\frac{\omega_2 x_4}{2} & -i\frac{x_1}{\omega_1} + \frac{x_3}{2} & -i\frac{x_2}{\omega_2} + \frac{x_4}{2} \\ y_1 - i\frac{\omega_1 y_3}{2} & -y_2 - i\frac{\omega_2 y_4}{2} & -i\frac{y_1}{\omega_1} + \frac{y_3}{2} & -i\frac{y_2}{\omega_2} + \frac{y_4}{2} \\ p_{x,1} - i\frac{\omega_1 p_{x,3}}{2} & -p_{x,2} - i\frac{\omega_2 p_{x,4}}{2} & -i\frac{p_{x,1}}{\omega_1} + \frac{p_{x,3}}{2} & -i\frac{p_{x,2}}{\omega_2} + \frac{p_{x,4}}{2} \\ p_{y,1} - i\frac{\omega_1 p_{y,3}}{2} & -p_{y,2} - i\frac{\omega_2 p_{y,4}}{2} & -i\frac{p_{y,1}}{\omega_1} + \frac{p_{y,3}}{2} & -i\frac{p_{y,2}}{\omega_2} + \frac{p_{y,4}}{2} \end{bmatrix} \tag{37}$$

$P_i = (2I_i\omega_i)^{1/2}\cos\phi_i, \quad Q_i = (\frac{2I_i}{\omega_i})^{1/2}\sin\phi_i, \quad (i = 1, 2)$ Under normality conditions:

$$x_1 p_{x,3} - x_3 p_{x,1} + y_1 p_{y,3} - y_3 p_{y,1} = 1$$
$$x_2 p_{x,4} - x_4 p_{x,4} + y_2 p_{y,4} - y_4 p_{y,2} = 1$$

Equivalently,

$$\begin{aligned} & -4i\omega_1 K_1 K_2\{\omega_1^2(F - E + n^2) \\ & + G^2 + 2E^2 + 3n^2E + n^2F - 2EF - 2n^4\} = 1 \end{aligned} \tag{38}$$

$$
\begin{aligned}
&- 4i\omega_2 K_2 K_4 \{\omega_2^2(F - E + n^2) \\
&+ G^2 + 2E^2 + 3n^2E + n^2F - 2EF - 2n^4\} = 1
\end{aligned}
\tag{39}
$$

$K_j's$ being arbitrary, we follow the approach of (Breakwell & Pringle 1966) and choose $J_{1,1} = J_{1,2} = 0$, implying that

$$
K_1(2in\omega_1 - G) = \frac{\omega_1 K_3}{2}(2n\omega_1 - iG), \quad K_2(G - 2in\omega_2) = \frac{\omega_2 K_4}{2}(2n\omega_2 - iG)
$$

i.e.

$$
\frac{K_1}{\omega_1(2n\omega_1 - iG)} = \frac{K_3}{2(2in\omega_1 - G)} = h_1 \text{ (say)} \tag{40}
$$

$$
\frac{K_2}{\omega_2(2n\omega_2 - iG)} = \frac{K_4}{2(G - 2in\omega_2)} = h_2 \text{ (say)} \tag{41}
$$

Using equations (38),(39),(40) and (41) we observe that

$$
h_j = \frac{1}{2\omega_j M_j \overline{M}_j (M_j^*)^2} \tag{42}
$$

where

$$
\begin{aligned}
&M_j = (\omega_j^2 - 2F + n^2)^{1/2}, \;\; M_j^* = (\omega_j^2 - 2E + n^2)^{1/2}, \\
&\overline{M_j} = \sqrt{2}(\omega_j^2 - E - 2 - n^2)^{1/2}, (j = 1, 2)
\end{aligned}
\tag{43}
$$

It is now verified that H_2 takes the form:

$$
H_2 = \frac{1}{2}(P_1^2 - P_2^2 + \omega_1^2 Q_1^2 - \omega_2^2 Q_2^2) \tag{44}
$$

We observe that

$$
J = [J_{ij}]_{4\times 4} = \begin{bmatrix}
0 & 0 & -\frac{M_1}{\omega_1 \overline{\overline{M_1}}} & \frac{iM_2}{\omega_2 \overline{\overline{M_2}}} \\
-\frac{2n\omega_1}{M_1 \overline{\overline{M_1}}} & \frac{2n\omega_2}{M_2 \overline{\overline{M_2}}} & \frac{G}{\omega_1 M_1 \overline{\overline{M_1}}} & \frac{iG}{\omega_2 M_2 \overline{\overline{M_2}}} \\
-\frac{\omega_1(m_1^2 - 2n^2)}{M_1 \overline{\overline{M_1}}} & \frac{i\omega_2(M_2^2 - 2n^2)}{M_2 \overline{\overline{M_2}}} & \frac{-nG}{\omega_1 M_1 \overline{\overline{M_1}}} & \frac{niG}{\omega_2 M_2 \overline{\overline{M_2}}} \\
\frac{-\omega_1 G}{M_1 \overline{\overline{M_1}}} & \frac{i\omega_1 G}{\omega_2 M_2 \overline{\overline{M_2}}} & -\frac{n(\omega_1^2 - M_1^2)}{\omega_1 M_1 \overline{\overline{M_1}}} & -\frac{ni(2\omega_2^2 - M_2^2)}{\omega_2 M_2 \overline{\overline{M_2}}}
\end{bmatrix} \tag{45}
$$

with $l_j^2 = 4\omega_j^2 + 9, (j = 1, 2)$ and $k_1^2 = 2\omega_1^2 - 1, k_2^2 = -2\omega_2^2 + 1$. Applying a contact transformation from Q_1, Q_2, P_1, P_2 to Q_1', Q_2', P_1', P_2' defined by (Whittaker 1965)

$$
P_j' = \frac{\partial W}{\partial Q_j}, \quad Q_j' = \frac{\partial W}{\partial P_j}, \quad (j = 1, 2)
$$

and

$$
W = \sum_{j=1}^{2} \left[Q_j' \sin^{-1}\left(\frac{P_j}{\sqrt{2\omega_j Q_j'}} \right) + \frac{P_j}{2\omega_j}\sqrt{2\omega_j Q_j' - P_j^2} \right]
$$

i.e.

$$
Q_j = \sqrt{\frac{2Q_j'}{\omega_j}} \cos P_j', \quad P_j = \sqrt{2\omega_j Q_j'} \sin P_j' \quad (j = 1, 2)
$$

The Hamiltonian H_2 is transformed into the form

$$H_2 = \omega_1 Q_1' - \omega_2 Q_2'$$

Denoting the angular variables P_1' and P_2' by ϕ_1 and ϕ_2 and the actions Q_1' and Q_2' by I_1, I_2 respectively, the second order part of the Hamiltonian transformed into the normal form

$$H_2 = \omega_1 I_1 - \omega_2 I_2 \tag{46}$$

The general solution of the corresponding equations of motion are

$$I_i = const., \quad \phi_i = \pm\omega_i + const, \ (i = 1, 2) \tag{47}$$

If the oscillations about L_4 are exactly linear, the equation(47) represent the integrals of motion and the corresponding orbits are given by

$$x = J_{13}\sqrt{2\omega_1 I_1}\cos\phi_1 + J_{14}\sqrt{2\omega_2 I_2}\cos\phi_2 \tag{48}$$

$$\begin{aligned} y = {} & J_{21}\sqrt{\frac{2I_1}{\omega_1}}\sin\phi_1 + J_{22}\sqrt{\frac{2I_2}{\omega_2}}\sin\phi_2 \\ & + J_{23}\sqrt{2I_1}\omega_1\cos\phi_1 + J_{24}\sqrt{2I_2}\omega_2\sin\phi_2 \end{aligned} \tag{49}$$

4. Conclusion

We have seen there are closed curves around the $L_{4(5)}$ so they are stable but the stability range reduced due to radiation effect. The effect of oblateness and mass of the belt is presented. We have found the normal form of the second order part of the Hamiltonian. For this we have solved the aforesaid set of equations. Under the normality conditions, we have applied a transformation defined by (Whittaker 1965). We have also utilized the approach of (Breakwell & Pringle 1966) for reducing the second order part of the Hamiltonian into the normal form. We have found that the second order part H_2 of the Hamiltonian is transformed into the normal form $H_2 = \omega_1 I_1 - \omega_2 I_2$.

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